

Case 5: $i > j$ and $j = 1, 1 \leq i \leq n - 1$

$$\alpha_{i,j} = -e_j \bar{c}_{n-i+1} \frac{\mu_1 \alpha^{n-2-i} + \mu_2 \beta^{n-2-i}}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})}. \quad (\text{A.11})$$

Case 6: $i > j$ and $j = 1, i = n$

$$\alpha_{i,j} = -e_j \frac{1}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})} \quad (\text{A.12})$$

Case 7: $i > j$ and $2 \leq j \leq n - 1, 1 \leq i \leq n - 1$

$$\alpha_{i,j} = -e_j \bar{c}_j \bar{c}_{n-i+1} \frac{(\mu_1 \alpha^{j-3} + \mu_2 \beta^{j-3})(\mu_1 \alpha^{n-2-i} + \mu_2 \beta^{n-2-i})}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})}. \quad (\text{A.13})$$

Case 8: $i > j$ and $2 \leq j \leq n - 1, i = n$

$$\alpha_{i,j} = -e_j \frac{(\mu_1 \alpha^{j-3} + \mu_2 \beta^{j-3})}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})}. \quad (\text{A.14})$$

For Cases 2 - 8, we can similarly obtain

$$|\alpha_{i,j}| \leq \frac{K_i}{\alpha^{|j-i|}}, \quad 2 \leq i \leq 8$$

where $K_2 \doteq 1.1726, K_3 \doteq 1.1607, K_4 \doteq 1.1666, K_5 \doteq 1.1666, K_6 \doteq 1.1607, K_7 \doteq 1.1722$ and $K_8 \doteq 1.1666$.

Finally, if we choose $K = 1.1726$, then

$$|\alpha_{i,j}| \leq \frac{K}{\alpha^{|j-i|}}, \quad 1 \leq i, j \leq n. \quad (\text{A.15})$$

This concludes the proof.

□

and

$$v_0 = -\frac{1}{a_1}(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4}).$$

where $\delta_1 = \frac{(13\alpha-1)\mu_1}{14} > 0, \delta_2 = \frac{(13\beta-1)\mu_2}{14} < 0$.

Finally, following (3.24) in Lemma 5, we have the following estimates on the inverse of M_j .

Denote $e_j, 1 \leq j \leq n$ as

$$e_j = \begin{cases} 1 & \text{if } j = 1 \\ \frac{13}{14} & \text{if } j = 2 \\ 1 & \text{if } 3 \leq j \leq n-1 \\ \frac{14}{13} & \text{if } j = n. \end{cases}$$

Case 1: $i \leq j$ and $1 \leq j \leq n-1, i = 1$

$$\alpha_{i,j} = -e_i \bar{c}_{n-j+1} \frac{(\mu_1 \alpha^{n-2-j} + \mu_2 \beta^{n-2-j})}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})} \quad (\text{A.7})$$

So we have

$$\begin{aligned} |\alpha_{i,j}| &\leq \frac{14}{13} \frac{\alpha^{n-2-j}(\mu_1 + \mu_2 z^{n-2-j})}{\alpha^{n-4}(\delta_1 - \delta_2 z^{n-4})} \\ &\leq \frac{14}{13} \frac{\alpha(\mu_1 + \mu_2/z)}{(\delta_1 - \delta_2)} \frac{1}{\alpha^{j-1}} \\ &= K_1 \frac{1}{\alpha^{|j-i|}} \end{aligned}$$

where $z = \frac{\beta}{\alpha} \leq 1$ and $K_1 \doteq 1.1666$.

Case 2: $i \leq j$ and $1 \leq j \leq n-1, 2 \leq i \leq n$

$$\alpha_{i,j} = -e_i \bar{c}_{n-j+1} \bar{c}_i \frac{(\mu_1 \alpha^{i-3} + \mu_2 \beta^{i-3})(\mu_1 \alpha^{n-2-j} + \mu_2 \beta^{n-2-j})}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})}. \quad (\text{A.8})$$

Case 3: $i \leq j$ and $j = n, i = 1$

$$\alpha_{i,j} = -e_i \frac{1}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})} \quad (\text{A.9})$$

Case 4: $i \leq j$ and $j = n, 2 \leq i \leq n$

$$\alpha_{i,j} = -e_i \bar{c}_i \frac{(\mu_1 \alpha^{i-3} + \mu_2 \beta^{i-3})}{(\delta_1 \alpha^{n-4} + \delta_2 \beta^{n-4})}. \quad (\text{A.10})$$

Appendix

Proof of (3.25). The proof is a straightforward application of **Lemma 5**. For M_j in (3.9), we have $(a_2, a_3, \dots, a_n) = (-\frac{1}{13}, -\frac{1}{14}, \dots, -\frac{1}{14}, -\frac{1}{14})$, $(b_1, b_2, \dots, b_n) = (1, 1, \dots, 1)$ and $(c_2, c_3, \dots, c_n) = (-\frac{1}{14}, -\frac{1}{14}, \dots, -\frac{1}{14}, -\frac{1}{13})$ where $n = n_j = 2^j L$. Therefore, the sequence $\{u_m\}$ in (3.22) satisfies the following relations,

$$u_0 = 0, u_1 = 1, u_2 = 14, u_3 = \frac{2534}{13}, \quad (\text{A.1})$$

and for $4 \leq m \leq n$

$$u_m = \bar{c}_m(-u_{m-2} + 14u_{m-1}) \quad (\text{A.2})$$

where $\bar{c}_m = \frac{13}{14}$, if $m = n$, $\bar{c}_m = 1$ otherwise.

Recursive relation (A.2) is a finite difference of order 2 whose general solution is of the following form

$$u_m = \bar{c}_m(c_1\alpha^{m-3} + c_2\beta^{m-3}) \quad (\text{A.3})$$

where $\alpha = 7 + \sqrt{192}/2, \beta = 7 - \sqrt{192}/2$ are the two distinct roots of the quadratic equation

$$x^2 - 14x + 1 = 0,$$

and constant c_1 and c_2 are chosen so equation (A.3) is valid for $m = 2, 3$.

Therefore,

$$\begin{aligned} u_0 &= 0, u_1 = 1 \\ u_m &= \bar{c}_m(\mu_1\alpha^{m-3} + \mu_2\beta^{m-3}), \quad 2 \leq m \leq n \end{aligned} \quad (\text{A.4})$$

where $\mu_1 = \frac{\alpha}{\alpha-\beta}(\frac{2534}{13} - 14\beta) > 0, \mu_2 = \frac{\beta}{\alpha-\beta}(14\alpha - \frac{2534}{13}) > 0$.

Similarly, we can show that

$$v_{n+1} = 0, v_n = 1, \quad (\text{A.5})$$

$$v_m = \bar{c}_{n-m+1}(\mu_1\alpha^{n-2-m} + \mu_2\beta^{n-2-m}), \quad \text{for } 1 \leq m \leq n-1 \quad (\text{A.6})$$

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Finally, we consider the IVB problem of nonlinear hyperbolic partial differential equation

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, & -1 \leq x \leq 2 \\ u(0, t) = \text{given} \\ u(x, 0) = f(x) \end{cases} \quad (6.5)$$

where

$$f(x) = \begin{cases} 7\sin(\pi x) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

In this case, we take $L = 10, J = 6$. Second order Runge-Kutta method is used for the time derivative. With every 10 iterations we change the number and locations of the collocation points according to the criteria proposed at the end of Section 5. The number of collocation points involved fluctuates around 100 in contrast to the full set collocation count which is 1280 in this case. The cut-off tolerance $\epsilon = 10^{-5}$. In Figure 14, we plot the numerical solutions at time $t = 0.05, 0.1$. The numerical scheme automatically puts more collocation points near the high gradient ($x=0$) and the derivative discontinuity ($x=1$).

7 Conclusion

In this paper, we have constructed a fast Discrete Wavelet Transform (DWT) which enables us to study collocation methods for nonlinear PDE's. The adaptivity of wavelet approximation is conveniently implemented through the examination of the wavelet coefficients. The preliminary tests on the solution of PDE's indicates such an approach will be important in large scale computation where the solution develops extremely high gradients in isolated regions, and uniform mesh is not practical. Such investigations are actually being done for reacting flows, the results will be reported in a separate paper.

Then, we use the procedure at the end of Section 5 to filter out those coefficients, thus deleting the corresponding collocation points, $\hat{f}_{j,k}$ which are less than ϵ in magnitude. In Figure 8, we take $\epsilon = 10^{-5}$ and the number of wavelet functions $\hat{f}_{j,k}$ reduced to 289 with the accuracy of the approximation (bottom curves) within order of ϵ . In Figure 9, we plot the solution at the remaining interpolation points and the expected clustering of the interpolation points is seen at locations where the function changes more dramatically. In Figure 10, we plot the magnitude of the wavelet coefficients $\hat{f}_{j,k}, j \geq -1$ one level above another. High density of the wavelet coefficients reflects the existence of high gradients of the approximated function. In Figure 11, we take $\epsilon = 10^{-4}$ and the number of wavelet functions $\hat{f}_{j,k}$ reduced to 206 with the accuracy of the approximation (bottom curves) within order of ϵ .

Linear Hyperbolic PDE's

We consider the IVB problem of linear hyperbolic partial differential equation

$$\begin{cases} u_t + u_x = 0, & 0 \leq x \leq 1 \\ u(0, t) = 0 \\ u(x, 0) = h_2(\frac{x}{2\delta}) \end{cases} \quad (6.4)$$

where $\delta = 0.05$ and $h_2(x)$ is defined in (6.3).

We apply the collocation method with adaptive choice of the collocation points $L = 20, J = 4$. Second order Runge-Kutta method is used for the time derivative. With every 10 iterations we change the number and locations of the collocation points according to the criteria proposed at the end of Section 5. The cut-off tolerance $\epsilon = 10^{-5}$. The number of collocation points involved fluctuates around 200 in contrast to the full set collocation count which is 640 in this case. In Figure 12, we plot the numerical solution ('+') against the exact solution ('o') at time $t = 0.1$. In Figure 13, we plot the errors in logarithm scale (notice the y-scale starts at -2 which corresponds to an error of 10^{-2}). Again, we see the automatic clustering of the collocation points.

Inviscid Burger Equation

We take the function in (6.1) and define its wavelet interpolation expansion (3.10) for $L = 10, J = 2, 3, \dots, 9$, the total number of terms (or collocation points) $N = 2^{J+1}L - 1$ are between 79 and 10240. In Figure 5 we plot the CPU time for the performance of DWT back and forth in both directions ('o' in the Figure) and the computations of derivatives on all collocation points ('+' in the Figure). Also drawn in the Figure is a straight line which indicates a almost linear growth of the CPU timing up to $10k$ points.

Adaptive Approximation of Wavelet Interpolation Expansion

We consider function

$$f(x) = \begin{cases} h_1(x+1, 0.3) & \text{if } -1 \leq x \leq -0.7 \\ 0 & \text{if } -0.7 \leq x \leq -0.5 - \delta \\ h_1(x+0.5, \delta) & \text{if } -0.5 - \delta \leq x \leq -0.5 + \delta \\ 0 & \text{if } -0.5 + \delta \leq x \leq 0 \\ \sin(5\pi x)h_1(x-0.25, 0, 25) & \text{if } 0 \leq x \leq 0.5 \\ h_2(\frac{x-0.5}{2\delta}) & \text{if } 0.5 \leq x \leq 1 \end{cases} \quad (6.1)$$

where $\delta = 0.01$ and $h_1(x, a)$ is an exponential hat function and $h_2(x)$ is a step-like function and they are defined as

$$h_1(x) = \begin{cases} \exp(-\frac{1}{a^2-x^2}) & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases} \quad (6.2)$$

and

$$h_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2772} \int_0^x t^5(1-t)^5 dt & \text{if } 0 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases} \quad (6.3)$$

First we construct the full wavelet interpolation expansion (3.10) $\mathcal{P}_J f(x)$ for $J = 6, L = 40$, the total number of wavelet functions (or the collocation points N) $N + 4 = (2^{J+1}L - 1) + 4 = 2^{J+1}L + 3 = 5123$ (including four boundary functions in $\mathbf{I}_{b,J}f(x)$). In Figure 6, on the top we plot the $f(x)$ (solid line) and $\mathcal{P}_J f(x)$ at non-interpolation points, at the bottom we have the absolute error in logarithm scale. In Figure 7, we plot the components $f_0 \in V_0$ and $g_j(x) \in W_j, 0 \leq j \leq 6$ in $\mathcal{P}_J f(x) = \mathbf{I}_{b,J}f(x) + f_0 + g_0 + \dots + g_J$. We can see that only higher frequency part is retained in higher wavelet spaces W_j (notice that the scales varies in different pictures).

Step 2. Following Theorem 3 and 4, we can ignore $\hat{u}_{j,k}$ in $u_j(x)$ in (5.2) for $k_i \leq k \leq l_i, i = 1, \dots, m, k_i = k'_i + \ell + 3, l_i = l'_i - \ell - 3$, namely we redefine $u_j(x)$ as

$$u_j(x) := \sum_{-1 \leq k \leq n_j - 2, k \notin \mathcal{K}_j} \hat{u}_{j,k} \psi_{j,k}(x)$$

where $\mathcal{K}_j = \bigcup_{1 \leq i \leq m} [k_i, l_i]$.

Step 3. The new collocation points and unknowns will be

$$\{x_k^{(j)}\}, u_J(x_k^{(j)}), k = 1, \dots, L - 1 \text{ if } j = -1; k \in \{-1, \dots, n_j - 2\} \setminus \mathcal{K}_j.$$

Increasing Level of Wavelet Space.

Let $\epsilon \geq 0$ again be some prescribed tolerance, and if

$$\max |\hat{u}_{J,k}^n| > \epsilon \tag{5.8}$$

where subscript n indicating the solution at time $t = t^n$, then we can increase the number of wavelet spaces W_j in the expansion for the numerical solution $u_J(x)$ in (5.2), say, up to $W_{J'}, J' > J$.

Step 1 At $t = t^n$ if condition (5.8) is satisfied, let $J' > J$ and define a new solution vector

$$\tilde{\mathbf{u}}_{J'}^n := (\mathbf{u}^{(-1)}, \mathbf{u}^{(0)}, \dots, \mathbf{u}^{(J)}, \mathbf{u}^{(J+1)}, \dots, \mathbf{u}^{(J')})^\top$$

where for $J + 1 \leq j \leq J'$, $\mathbf{u}^{(j)} = \{\mathcal{P}_J \mathbf{u}(x_k^{(j)})\}_{k=-1}^{n_j-2}$.

Step 2 Use $\tilde{\mathbf{u}}_{J'}^n$ on the right hand side of scheme (5.5) to advance the solution to time step t^{n+1} and obtain solution $\mathbf{u}_{J'}^{n+1}$. Then, $u_{J'}^{n+1}(x) = \mathcal{P}_{J'} u_{J'}^{n+1} \in V_0 \oplus W_0 \oplus \dots \oplus W_{J'}$ will be the new numerical solution which yields better approximation to the exact solution of (5.1).

6 Numerical Results

CPU Performance of DWT transform

The theoretical estimates of operations for performing the DWT transform in both direction and the computation of derivatives at all collocation points are $O(N \log N)$ where N is the total number of terms in the wavelet expansion (3.10).

Again, $\psi'(x)$ and $\phi'(x)$ at the dyadic points $\frac{k}{2^j}$, $0 \leq k \leq 2^j L$ can be precomputed once and for all.

Assuming that Euler forward is used to discretize the time derivative in (5.3), we obtain a fully discretized wavelet collocation method.

Fully discretized Wavelet Collocation Method

$$\begin{cases} u_J^{n+1} &= u_J^n + \Delta t [-f_x(u_J^n) + u_{J_{xx}}^n + g(u_J^n)]|_{x=x_k^{(j)}}, -1 \leq j \leq J \\ u_J^n(0) &= g_0(t^n) \\ u_J^n(L) &= g_1(t^n) \\ u_J^n(x = x_k^{(j)}) &= f(x = x_k^{(j)}) \end{cases} \quad (5.5)$$

where $1 \leq k \leq L - 1$ for $j = -1$; $-1 \leq k \leq n_j - 2$, for $j \geq 0$ and $t^n = n\Delta t$ is the time station and Δt is the time step.

Adaptive Choice of Collocation Points

In equations (5.2) and (5.4), $u_J(x)$ and $f(u_J(x))$ are expressed using the full set of collocation points $\{x_k^{(j)}\}$. As discussed in the remark after Theorem 3 of Section 3, most of the wavelet expansion coefficients $\hat{u}_{j,k}$ for large j can be ignored within a given tolerance ϵ . So we can dynamically adjust the number and locations of the collocation points used in the wavelet expansions, thus reducing significantly the cost of the scheme while providing enough resolution in the regions where solution varies much. We can achieve this adaptivity in the following two ways.

Deleting Collocation Points

Let $\epsilon \geq 0$ be a prescribed tolerance and $j \geq 0$, $\ell = \ell(\epsilon) = \min(\frac{n_j}{2}, -\log \epsilon / \log \alpha)$.

Step 1. First we locate the range for the index k ,

$$(k'_1, l'_1), \dots, (k'_m, l'_m), m = m(j, \epsilon) \quad (5.6)$$

such that

$$|\hat{u}_{j,k}| \leq \epsilon, \quad k'_i \leq k \leq l'_i, \quad i = 1, \dots, m. \quad (5.7)$$

Equation (5.3) involves total $(2^{J+1} - 1)L + 2$ unknowns in \mathbf{u} two of which will be determined by the boundary conditions and the rest are the solutions of the ODE system subject to their initial conditions. In order to implement the time marching scheme for the ODE's system (for example Runge-Kutta type time integrator), we have to compute the derivative term in (5.3) $f_x(u_J(x_k^{(j)}))$ and $u_{Jxx}(x_k^{(j)})$ in an efficient way. Let us only discuss the first derivative which involves the computation of the nonlinear function $f(u_J(x, t))$. For this purpose we first find a similar wavelet decomposition as (5.2) for $f(u_J)$. For a general nonlinear function $f(u)$, this can be done quite straightforward using the DWT transform in Section 3.

Computation of $f_x((x_k^{(j)})) = f_x(u_J(x_k^{(j)}))$

Step 1 Given $\mathbf{u} = (\mathbf{u}^{(-1)}, \mathbf{u}^{(0)}, \dots, \mathbf{u}^{(J)})^\top$, compute $\mathbf{f}^{(j)} = \{f(u_k^{(j)})\}$, $j \geq -1$ and define

$$\mathbf{f} = (\mathbf{f}^{(-1)}, \mathbf{f}^{(0)}, \dots, \mathbf{f}^{(J)})^\top;$$

Step 2 Compute the wavelet interpolation expansion using DWT transform for \mathbf{f} ,

$$\begin{aligned} f_J(x, t) &= \mathbf{I}_{b,J} f + \hat{f}_{-1,-1}(t) \phi_b(x) + \sum_{k=0}^{L-4} \hat{f}_{-1,k}(t) \phi_k(x) + \hat{f}_{-1,L-3}(t) \phi_b(L - x) \\ &+ \sum_{j=0}^J \left[\sum_{k=-1}^{n_j-2} \hat{f}_{j,k}(t) \psi_{j,k}(x) \right]; \end{aligned} \quad (5.4)$$

Step 3 Differentiate (5.4) and evaluate at all collocation points $\{x_k^{(j)}\}, j \geq -1$,

$$\begin{aligned} f_x(u_J)|_{x=x_k^{(j)}} &= (\mathbf{I}_{b,J} f)'(x_k^{(j)}) + \hat{f}_{-1,-1}(t) \phi'_b(x_k^{(j)}) + \sum_{k=0}^{L-4} \hat{f}_{-1,k}(t) \phi'_k(x_k^{(j)}) - \hat{f}_{-1,L-3}(t) \phi'_b(L - x_k^{(j)}) \\ &+ \sum_{i=0}^J \left[\sum_{l=-1}^{n_i-2} \hat{f}_{i,l}(t) \psi'_{i,l}(x_k^{(j)}) \right]. \end{aligned}$$

Cost of Computing the Derivatives.

For each single collocation point, it takes $7 + 5(J + 1) = 5J + 12$ (flops) to compute $f'_J(x_k^{(J)})$. Therefore, the total cost of computing all derivatives is $(5J + 12)N \leq 5N \log N$.

value problem

$$\begin{cases} u_t + f_x(u) &= u_{xx} + g(u), x \in [0, L], t \geq 0 \\ u(0, t) &= g_0(t) \\ u(L, t) &= g_1(t) \\ u(x, 0) &= f(x) \end{cases} \quad (5.1)$$

where only Dirichlet boundary conditions are considered, however, the methods presented here can also be modified to treat Von Neuman type or Robin type boundary conditions.

We use the idea of method of lines where only the spatial derivative is discretized by the wavelet decomposition. The numerical solution $u_J(x, t)$ will be represented by an unique decomposition in $V_0 \oplus W_0 \oplus \cdots \oplus W_J, J \geq 0$, namely

$$\begin{aligned} u_J(x, t) &= \mathbf{I}_{b,J}u(x, t) + \hat{u}_{-1,-1}(t)\phi_b(x) + \sum_{k=0}^{L-4} \hat{u}_{-1,k}(t)\phi_k(x) + \hat{u}_{-1,L-3}(t)\phi_b(L-x) \\ &+ \sum_{j=0}^J \left[\sum_{k=-1}^{n_j-2} \hat{u}_{j,k}(t)\psi_{j,k}(x) \right], \\ &= u_{-1}(x) + \sum_{j=0}^J u_j(x) \end{aligned} \quad (5.2)$$

where $\mathbf{I}_{b,J}u(x, t)$ given in (2.33) consists the nonhomogeneity of $u(x, t)$ on both boundaries, and the coefficients $\hat{u}_{j,k}(t)$ are all functions of t . Using the DWT transform, we can also identify the numerical solution $u_J(x, t)$ by its point values on all collocation (previously named interpolation) points, i.e. $\{x_k^{(j)}\}$ in (3.1) and (3.6), we put all these values in vector $\mathbf{u} = \mathbf{u}(t)$, i.e.

$$\mathbf{u} = \mathbf{u}(t) = (\mathbf{u}^{(-1)}, \mathbf{u}^{(0)}, \dots, \mathbf{u}^{(J)})^\top$$

where $\mathbf{u}^{(j)} = \{u(x_k^{(j)}, t)\}, 1 \leq k \leq L-1$ for $j = -1; -1 \leq k \leq n_j - 2$, for $j \geq 0$.

To solve for the unknown solution vector $\mathbf{u}(t)$, we collocate the PDE (5.1) on all collocation points, then we have the following semi-discretized wavelet collocation method.

Semi-Discretized Wavelet Collocation Methods

$$\begin{cases} u_{J,t} + f_x(u_J) &= u_{J,xx} + g(u_J)|_{x=x_k^{(j)}}, -1 \leq j \leq J \\ u_J(0, t) &= g_0(t) \\ u_J(L, t) &= g_1(t) \\ u_J(x = x_k^{(j)}, 0) &= f(x = x_k^{(j)}) \end{cases} \quad (5.3)$$

where $1 \leq k \leq L-1$ for $j = -1; -1 \leq k \leq n_j - 2$, for $j \geq 0$

$$\begin{aligned}
&= H\mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{u}' \\
&= (0, H\mathbf{B}^{-1}\mathbf{\Gamma}, 0)\mathbf{u}
\end{aligned} \tag{4.9}$$

Finally, combining equations (4.8) and (4.9), we have

$$\mathbf{u}_\xi = \begin{pmatrix} D_\xi u_J(0) \\ D_\xi u_J(1) \\ \vdots \\ D_\xi u_J(L') \end{pmatrix} = \mathcal{D}'\mathbf{u} \tag{4.10}$$

where the derivative matrix \mathcal{D}' is defined by

$$\mathcal{D}' = \Delta + (0, H\mathbf{B}^{-1}\mathbf{\Gamma}, 0). \tag{4.11}$$

Converting to the x -derivatives, we have

$$\mathbf{u}_x = \begin{pmatrix} D_x u_J(x_0) \\ D_x u_J(x_1) \\ \vdots \\ D_x u_J(x_{L'}) \end{pmatrix} = 2^{J+1} \begin{pmatrix} D_\xi u_J(0) \\ D_\xi u_J(1) \\ \vdots \\ D_\xi u_J(L') \end{pmatrix} = 2^{J+1} \mathcal{D}'\mathbf{u} \tag{4.12}$$

where $x_i = \frac{i}{2^{J+1}}$, $0 \leq i \leq L'$.

Let \mathcal{D} be the upper left $L' \times L'$ submatrix of \mathcal{D}' , $\frac{1}{2^{J+1}}\mathcal{D}$ will be the wavelet derivative matrix to differential operator (4.1) with boundary condition (4.2), namely, \mathcal{D} will maps the function values $u(x_0), u(x_1), \dots, u(x_{L'-1})$ to its derivatives $u'(x_0), u'(x_1), \dots, u'(x_{L'-1})$,

$$\begin{pmatrix} u'(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_{L'-1}) \end{pmatrix} = 2^{J+1} \mathcal{D} \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{L'-1}) \end{pmatrix}. \tag{4.13}$$

In Figure 15, we plot the eigenvalues of \mathcal{D} for $L = 8, J = 0, 1, 2, 3$ which corresponds to $N = 8, 16, 32, 64$. The eigenvalues come in conjugate pairs with two pure real eigenvalues. The real part of all the eigenvalues are negative and except one eigenvalues, all the rest are close the imaginary axis.

5 Adaptive Wavelet Collocation Methods for PDE's

In this section we consider a collocation method based on the DWT transform given in Section 3 for time dependent PDE's. Let $u = u(x, t)$ be the solution of the following initial

where \mathbf{I} is the $(L' - 1) \times (L' - 1)$ identity matrix.

To obtain approximation to the derivatives of $u(\xi)$, we differentiate equation (4.4) with respect to ξ and evaluate at $\xi_k = k, 0 \leq k \leq L'$, i.e.

$$\begin{aligned} u'_J(\xi_k) &= (\mathbf{I}_{b,0}u)'(\xi_k) + \hat{u}_{-1}\phi'_b(\xi) + \sum_{k=0}^{L'-4} \hat{u}_k\phi'_k(\xi_k) - \hat{u}_{L'-3}\phi'_b(L' - \xi_k) \\ &= u'_1(\xi_k) + u'_2(\xi_k) \quad 0 \leq k \leq L' \end{aligned} \quad (4.7)$$

where $u'_1(\xi_k)$ denotes the first term in the first equation and $u'_2(\xi_k)$ the rest. Recalling the definition of $\mathbf{I}_{b,0}u(\xi)$ in (2.33) and coefficients α_k in (2.38) with $h = 1$ and $j = J + 1, L = L'$, we have

$$\begin{pmatrix} u'_1(0) \\ u'_1(1) \\ \vdots \\ u'_1(L') \end{pmatrix} = \begin{pmatrix} 2\sum_{k=0}^p c'_k u(k) - 3u(0) \\ -\frac{1}{2}\sum_{k=0}^p c'_k u(k) \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\sum_{k=0}^p c'_k u(L' - k) \\ -2\sum_{k=0}^p c'_k u(L' - k) + 3u(L') \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ 0 \\ \vdots \\ 0 \\ \delta_3 \\ \delta_4 \end{pmatrix} \mathbf{u} = \mathbf{\Delta} \mathbf{u} \quad (4.8)$$

with the four $L' + 1$ dimension vectors

$$\begin{aligned} \delta_1 &= (2c'_0 - 3, 2c'_1, 2c'_2, 2c'_3, 0, \dots, 0) \in \mathbf{R}^{L'+1} \\ \delta_2 &= -\frac{1}{2}(c'_0, c'_1, c'_2, c'_3, 0, \dots, 0) \in \mathbf{R}^{L'+1} \\ \delta_3 &= \frac{1}{2}(0, \dots, 0, c'_3, c'_2, c'_1, c'_0) \in \mathbf{R}^{L'+1} \\ \delta_4 &= -(0, \dots, 0, 2c'_3, 2c'_2, 2c'_1, 2c'_0 - 3) \in \mathbf{R}^{L'+1}. \end{aligned}$$

On the other hand, using (3.5) we have

$$\begin{aligned} \begin{pmatrix} u'_2(0) \\ u'_2(1) \\ \vdots \\ u'_2(L') \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & & & \\ 0 & -\frac{1}{2} & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & & & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \hat{\mathbf{u}} \\ &= H \hat{\mathbf{u}} \end{aligned}$$

Because of the multiresolution structure of spaces V_j , i.e. $V_j \subset V_{j+1}$ and $V_0 \oplus W_0 \oplus \cdots \oplus W_J = V_{J+1}$. We can rewrite the wavelet interpolation $u_J(x)$ of (2.39) for function $u(x)$ as a linear combination of $\mathbf{I}_{b,J+1}f(x)$ and basis in V_{J+1} , namely

$$u_J(x) = \mathbf{I}_{b,J+1}u(x) + \hat{u}_{-1}\phi_{b,J+1}(x) + \sum_{k=0}^{L'-4} \hat{u}_k\phi_{J+1,k}(x) + \hat{u}_{L'-3}\phi_{b,J+1}(L-x) \quad (4.3)$$

where $L' = 2^{J+1}L$ and $\mathbf{I}_{b,J+1}u(x)$ is defined in (2.33).

With the transformation $\xi = 2^{J+1}x$, equation (4.3) becomes

$$u_J(\xi) := u_J(x) = \mathbf{I}_{b,0}u(\xi) + \hat{u}_{-1}\phi_b(\xi) + \sum_{k=0}^{L'-4} \hat{u}_k\phi_k(\xi) + \hat{u}_{L'-3}\phi_b(L' - \xi). \quad (4.4)$$

Using the notations

$$\mathbf{u}' = (u(1), u(2), \dots, u(L' - 1))^\top \in \mathbf{R}^{L'-1},$$

$$\mathbf{u} = (u(0), (\mathbf{u}')^\top, u(L'))^\top \in \mathbf{R}^{L'+1}$$

$$\hat{\mathbf{u}} = (\hat{u}_{-1}, \hat{u}_0, \dots, \hat{u}_{L'-3})^\top \in \mathbf{R}^{L'-1},$$

and equation (3.5), we have

$$\hat{\mathbf{u}} = \mathbf{B}^{-1}(\mathbf{u}' - \mathbf{u}_b) \quad (4.5)$$

where vector \mathbf{u}_b is defined by

$$\mathbf{u}_b = (\mathbf{I}_{b,0}u(1), 0, \dots, 0, \mathbf{I}_{b,0}u(L' - 1))^\top \in \mathbf{R}^{L'-1}$$

and

$$\begin{aligned} \mathbf{I}_{b,0}u(1) &= \frac{1}{6}(c'_0, c'_1, c'_2, c'_3, 0, \dots, 0)\mathbf{u}' = \gamma_1\mathbf{u}', \quad \gamma_1 \in \mathbf{R}^{L'-1} \\ \mathbf{I}_{b,0}u(L' - 1) &= \frac{1}{6}(0, \dots, 0, -c'_3, -c'_2, -c'_1, -c'_0) = \gamma_2\mathbf{u}', \quad \gamma_2 \in \mathbf{R}^{L'-1}. \end{aligned}$$

Therefore,

$$\hat{\mathbf{u}} = \mathbf{B}^{-1}(\mathbf{I} - \begin{bmatrix} \gamma_1 \\ 0 \\ \vdots \\ 0 \\ \gamma_2 \end{bmatrix})\mathbf{u}' = \mathbf{B}^{-1}\Gamma\mathbf{u}' \quad (4.6)$$

used successfully in the shock wave computations with uniform high order spectral methods, where ENO finite difference methods and spectral methods are combined to resolve the shocks and the high frequency components in the solution, respectively [17].

We conclude this section with the following result which shows how to use wavelet coefficient to estimate the data interpolated by \mathbf{I}_{w_j} .

Theorem 4 Let $\mathbf{I}_{w_j}f(x)$ and $f(x)$ as in Theorem 4. And if for $\epsilon > 0$, $-1 \leq k_1 < k_2 \leq n_j - 2$

$$|\hat{f}_{j,k}| \leq \epsilon \text{ for } k_1 \leq k \leq k_2,$$

then

$$|f(x_k^{(j)})| \leq 3\epsilon \text{ for } k_1 + 3 \leq k \leq k_2 - 3. \quad (3.29)$$

Proof. The proof follows from the definition of $\mathbf{I}_{w_j}f(x)$.

□

4 Derivative Matrix \mathcal{D}

The operation of differentiation of functions, which are given in terms of the wavelet expansion of (2.39), can be represented by a finite dimension matrix \mathcal{D} . Such matrix has been investigated in [16] for wavelet approximation based on Daubechie's compactly supported wavelets for periodic functions. The properties of matrix \mathcal{D} , especially of its eigenvalues, affect very much the efficiency and stability of the numerical methods for the solution of PDE's to be discussed in the next section.

We consider the derivative matrix which approximates the first differential operator

$$\mathcal{L}u = u_x \quad (4.1)$$

with the boundary condition

$$u(L) = 0. \quad (4.2)$$

$$\begin{aligned}
&\leq K\epsilon \sum_{|k-i|\leq\ell} \frac{1}{\alpha^{|k-i|}} + MK \sum_{|k-i|>\ell} \frac{1}{\alpha^{|k-i|}} \\
&\leq 2K\epsilon[1 + (\frac{1}{\alpha}) + \cdots + (\frac{1}{\alpha})^\ell] + 2MK[(\frac{1}{\alpha})^{\ell+1} + \cdots + (\frac{1}{\alpha})^{n_j-\ell}] \\
&= 2K\epsilon \frac{1 - (\frac{1}{\alpha})^{\ell+1}}{1 - (\frac{1}{\alpha})} + 2MK(\frac{1}{\alpha})^{\ell+1} \frac{1 - (\frac{1}{\alpha})^{n_j-2\ell}}{1 - (\frac{1}{\alpha})} \\
&\leq 2K\epsilon \frac{\alpha}{\alpha-1} + 2MK\epsilon \frac{1}{\alpha-1} \\
&= C'\epsilon
\end{aligned}$$

where $C' = \frac{2K}{\alpha-1}(\alpha + M)$.

Finally, we have

$$\begin{aligned}
|\tilde{\mathbf{I}}_{w_j}f(x) - \mathbf{I}_{w_j}f(x)| &= \left| \sum_{k \in [k_1+\ell, k_2-\ell]} \hat{f}_{j,k} \psi_{j,k}(x) \right| \\
&\leq \sum_{k \in [k_1+\ell, k_2-\ell]} |\hat{f}_{j,k}| |\psi_{j,k}(x)| \leq C'\epsilon \sum_{k \in [k_1+\ell, k_2-\ell]} |\psi_{j,k}(x)|
\end{aligned}$$

Note that in the last summation, only three terms will be nonezero for any fixed x , so we have

$$|\tilde{\mathbf{I}}_{w_j}f(x) - \mathbf{I}_{w_j}f(x)| \leq 3C'\epsilon = C\epsilon$$

where $C = \frac{6K}{\alpha-1}(\alpha + M)$. This concludes the proof of the Theorem.

□

Remark. As a consequence of Theorem 3, the coefficients $\hat{f}_{j,k}$ of the wavelet interpolation operator $\mathbf{I}_{w_j}f(x)$ can be ignored if $x_k^{(j)} \in [x_{k_1+\ell}^{(j)}, x_{k_2-\ell}^{(j)}]$ where the function $f(x)$ is less than some given error tolerance ϵ . This procedure will only result in an error of $O(\epsilon)$. For $\epsilon = 10^{-10}, \ell = 9, \epsilon = 10^{-8}, \ell = 7$. In the wavelet interpolation expansion (3.10), \mathbf{I}_{w_j} is used to interpolate the difference between a lower level interpolation $\mathcal{P}_{j-1}f(x)$ and $f(x)$, i.e. $\mathcal{P}_{j-1}f(x) - f(x)$. Thus, the situation mentioned here will occur in larger region of the solution domain as j becomes larger, avoiding adding unnecessary expansion terms $\psi_{j,k}(x)$. This fact will be used in the later section to achieve adaptivity for the solution of PDE's. The idea of decomposing numerical approximations into different scales has been previously

Corollary. Let M_j be the interpolation matrix in (3.9), then we have the following estimates on $M_j^{-1} = (\alpha_{i,j})$,

$$|\alpha_{i,j}| \leq \frac{K}{\alpha^{|j-i|}} \quad (3.25)$$

where $K = 1.1726$ and $\alpha = 7 + \sqrt{192} \doteq 13.928$.

We delay the proof of (3.25) to the Appendix.

Theorem 3 Let $f(x) \in H_0^2(0, L)$ and $M = \max_I |f(x)|$ and $\mathbf{I}_{w_j} f(x)$ be its interpolation in W_j defined in (3.8) and if for $\epsilon > 0$, $-1 \leq k_1 < k_2 \leq n_j - 2$

$$|f(x_k^{(j)})| \leq \epsilon \quad \text{for } k_1 \leq k \leq k_2.$$

then define

$$\tilde{\mathbf{I}}_{w_j} f(x) = \sum_{-1 \leq k \leq n_j - 2, k \in [k_1 + l, k_2 - l]} \hat{f}_{j,k} \psi_{j,k}(x), \quad (3.26)$$

where $l = l(\epsilon) = \min(\frac{n_j}{2}, -\frac{\log \epsilon}{\log \alpha})$. We have

$$|\tilde{\mathbf{I}}_{w_j} f(x) - \mathbf{I}_{w_j} f(x)| \leq C(M)\epsilon \quad (3.27)$$

where $C(M) = \frac{6K}{\alpha-1}(\alpha + M)$, $K = 1.1726$ and $\alpha = 7 + \sqrt{192} \doteq 13.928$.

Proof. From (3.9), we have

$$\hat{\mathbf{f}}^{(j)} = \mathbf{M}_j^{-1} \mathbf{f}^{(j)}$$

where $\hat{\mathbf{f}}^{(j)} = (\hat{f}_{j,-1}, \dots, \hat{f}_{j,n_j-2})^\top$, $\mathbf{f}^{(j)} = (f(x_{-1}^{(j)}), \dots, f(x_{n_j-2}^{(j)}))^\top$, thus

$$\hat{f}_{j,k} = \sum_{i=1}^{n_j} \alpha_{k,i} f(x_{i-2}^{(j)}), \quad -1 \leq k \leq n_j - 2.$$

So we have

$$|\hat{f}_{j,k}| \leq K \sum_{i=1}^{n_j} \frac{1}{\alpha^{|k-i|}} |f(x_{i-2}^{(j)})|. \quad (3.28)$$

For any given $\epsilon > 0$, we take $\ell = \min(n_j/2, -\log \epsilon / \log \alpha)$. For $k \in [k_1 + \ell, k_2 - \ell]$, using (3.28) we have

$$|\hat{f}_{j,k}| \leq K \left[\sum_{|k-i| \leq \ell} \frac{1}{\alpha^{|k-i|}} |f(x_{i-2}^{(j)})| + \sum_{|k-i| > \ell} \frac{1}{\alpha^{|k-i|}} |f(x_{i-2}^{(j)})| \right]$$

for some $\delta > 0$, then for any $k \in Z, j \in Z^+$ such that $2^{-j}k \in (x_0 - \delta/2, x_0 + \delta/2)$,

$$|\hat{f}_{j,k}| = O(2^{-(\alpha+1)j}), \quad \text{as } j \rightarrow \infty. \quad (3.21)$$

Proof. We have, since ψ and ψ_b defined by (2.19) and (2.20) are continuous and their second derivatives have 2 vanishing moments, by Theorem A,

$$\begin{aligned} |\hat{f}_{j,k}| &= | \langle f, \psi_{jk}^* \rangle | \leq \sum_{2^{-j}l \in (x_0 - \delta, x_0 + \delta)} |\alpha_{kl}^{(j)}| | \langle f, \psi_{jl} \rangle | + \sum_{2^{-j}l \notin (x_0 - \delta, x_0 + \delta)} |\alpha_{kl}^{(j)}| | \langle f, \psi_{jl} \rangle | \\ &\leq \sum_{2^{-j}l \in (x_0 - \delta, x_0 + \delta)} K \lambda^{|k-l|} O(2^{-(\alpha+1)j}) + \sum_{|l-k| > 2^j \frac{\delta}{2}} K \lambda^{|l-k|} C = O(2^{-(\alpha+1)j}) \end{aligned}$$

where C in the first but last equation is a constant which depends on the second derivative of $f(x)$.

□

Lemma 4 implies that the wavelet coefficients $\hat{f}_{jk}, j \geq 0$, still reflect the singularity of the function to be approximated. In practice, when we solve PDE's using collocation methods, we often use the values of the functions, not their derivatives. Therefore, in order to use the wavelet coefficients to adjust the choice of wavelet basis functions, we have to establish a relation between the magnitude of the wavelet coefficients $\hat{f}_{j,k}, j \geq 0$ and $f(x)$. Let us first state the following result on the inverse of tridiagonal matrix from [15].

Lemma 5 Let A be a $n \times n$ tridiagonal matrix with elements a_2, a_3, \dots, a_n on the subdiagonal, b_1, b_2, \dots, b_n on the diagonal and c_2, c_3, \dots, c_n on the superdiagonal, where $a_i, c_i \neq 0$. Define the two sequence $\{u_m\}, \{v_m\}$ as follows:

$$u_0 = 0, u_1 = 1, u_m = -\frac{1}{c_m}(a_{m-1}u_{m-2} + b_{m-1}u_{m-1}) \quad m \geq 2 \quad (3.22)$$

$$v_{n+1} = 0, v_n = 1, v_m = -\frac{1}{a_{m+1}}(b_{m+1}v_{m+1} + c_{m+2}v_{m+1}) \quad m \leq n-1 \quad (3.23)$$

where a_1 and c_{n+1} are arbitrary nonzero constants. Then $A^{-1} = (\alpha_{i,j})$ is given by

$$\alpha_{i,j} = \begin{cases} -\frac{u_i v_j}{a_1 v_0} \prod \frac{c_k}{a_k} & i \leq j \\ -\frac{u_j v_i}{a_1 v_0} \prod \frac{c_k}{a_k} & i > j \end{cases} \quad (3.24)$$

From Theorem A, we can claim that the absolute value of the wavelet coefficient $|f_{jk}|$ depends upon the local regularity of $f(x)$ in the neighborhood of the abscissa $2^{-j}k$. More precisely, if $2^{-j}k \in (a, b)$, the decay of $|f_{jk}|$ depends upon the Lipschitz regularity of $f(x)$ over the interval $[a, b]$, as the resolution 2^j increases. This property of the wavelet coefficients allow us to detect the location of the singularity of the function and, then, provide a general knowledge of the distribution of wavelet basis functions whose coefficients are larger in magnitude than a given threshold. The detail can be referred to [11]. In the framework of L^2 , the wavelet function $\psi(x)$ has at least 1 vanishing moment. Hence the property of the wavelet coefficients mentioned above is always valid.

Now we return to the wavelet coefficients $\{\hat{f}_{jk}\}$ in (3.10). It can be easily checked that

$$\hat{\psi}(\omega) = \frac{7}{3}(2 - \cos \omega) \left(\frac{\sin \frac{\omega}{4}}{\frac{\omega}{4}}\right)^4 e^{-\frac{3\omega}{2}i}.$$

Hence $\hat{\psi}(0) = \frac{3}{7}$, which implies that ψ has no vanishing moment at all (see Figure 3). Since the wavelet decomposition we considered here is in the space $H_0^2(I)$, therefore, the decay property for the wavelet coefficients \hat{f}_{jk} ought to be related to the vanishing moments of the second derivative of $\psi(x)$ (see Figure 4), not to those of $\psi(x)$. We shall illustrate this more precisely.

Let $\{\psi_{jk}^*\}$ be the dual basis of $\{\psi_{jk}\}$ in W_J (recalling that the space we consider here is H_0^2). It can be proven that for Ψ_{jk}^* , (3.17) and (3.18) still hold. Then we have

$$\hat{f}_{jk} = \int_I f''(x)(\psi_{jk}^*)''(x) dx. \quad (3.20)$$

Notice that spline wavelets $\psi(x)$ and $\psi_b(x)$ defined by (2.19) and (2.20) are continuous and their second derivatives have 2 vanishing moments. Then applying Theorem A to the second derivatives of $\psi(x)$ and $\psi_b(x)$ (namely, by taking $g(x)$ in Theorem A to be $\psi''(x)$ and $\psi_b''(x)$ respectively), we can prove the following.

Lemma 4 Let $0 < \alpha < 1$ and $f \in H_0^2(I)$. If the second derivative of the function f is Hölder continuous with exponent α , at $x_0 \in I$, i.e.

$$|f''(x) - f''(x_0)| \leq C|x - x_0|^\alpha, x \in (x_0 - \delta, x_0 + \delta) \subset I$$

a look at the wavelet coefficient in the finite wavelet decomposition of space $L^2(I)$, i.e. in the decomposition $V_j = V_0 \oplus W_0 \oplus W_1 \cdots \oplus W_j$. The orthogonality here is in the sense of L^2 norm not of $H_0^2(I)$ norm. For simplicity, we still use the notation $\phi(x)$ and $\psi(x)$ to denote the scaling function and the wavelet function for this decomposition, while keeping in mind that they have different definitions from those of $H_0^2(I)$. Then, we can write the wavelet coefficient f_{jk} in the finite decomposition as,

$$f_{jk} = \int_I f(x) \psi_{jk}^*(x) dx$$

where $\{\psi_{jk}^*\}$ is the dual wavelet basis of $\{\psi_{jk}\}$ in W_j . i.e. $\{\psi_{jk}^*\}$ is such a basis of W_j that

$$\int \psi_{jk}^*(x) \psi_{jl}(x) dx = \delta_{lk}$$

where δ_{lk} is the Kronecker symbol.

Using a similar method in [12], we can prove that

$$\psi_{jk}^* = \sum_{l=1}^{n_j} \alpha_{kl}^{(j)} \psi_{jl} \quad (3.17)$$

where $\alpha_{kl}^{(j)}$ satisfies the estimate

$$|\alpha_{kl}^{(j)}| \leq K \lambda^{|l-k|} \quad (3.18)$$

with $0 < \lambda < 1$ and K a constant.

In order to estimate $\int f \psi$, we quote the following theorem from Meyer's book [9].

Theorem A Let $g(x)$ be compactly supported, n times continuously differentiable and have $n+1$ vanishing moments:

$$\int_{-\infty}^{\infty} x^p g(x) dx = 0, \text{ for } 0 \leq p \leq n.$$

Let $\alpha, 0 < \alpha < n$, be a real number that is not an integer and $f(x) \in L^2$. Then $f(x)$ is uniformly Lipschitz of order α over a finite interval $[a, b]$ if and only if for any $k \in Z$ and $\triangleright \in Z$ such that $2^{-j}k \in (a, b)$,

$$\left| \int f(x) g(2^j x - k) dx \right| = O(2^{-(\alpha+1)j}) \quad \text{as } j \rightarrow \infty. \quad (3.19)$$

So we have

$$\begin{aligned} f_{-1}(x_k^{(-1)}) + f_0(x_k^{(-1)}) &= f_{-1}(x_k^{(-1)}) = \mathbf{I}_{V_0}f(x_k^{(-1)}) = f(x_k^{(-1)}) \quad 1 \leq k \leq L-1 \\ f_{-1}(x_k^{(0)}) + f_0(x_k^{(0)}) &= \mathbf{I}_{V_0}f(x_k^{(0)}) + (\mathbf{f}_k^{(0)} - (\mathbf{I}_{V_0}f)_k^{(0)}) = \mathbf{f}_k^{(0)} = f(x_k^{(0)}). \end{aligned} \quad (3.14)$$

Equation (3.14) implies that function $f_{-1}(x) + f_0(x)$ actually interpolates $f(x)$ on both interpolation points $\{x_k^{(-1)}\}$ for V_0 and the interpolation points $\{x_k^{(0)}\}$ for W_0 .

Step 3

Generally, we define for $1 \leq j \leq J$

$$f_j(x) = \mathbf{I}_{w_j}(\mathbf{f}^{(j)} - (\mathcal{P}_{j-1}f)^{(j)}) \quad (3.15)$$

$$= \sum_{k=-1}^{n_j-2} \hat{f}_{j,k} \psi_{j,k}(x). \quad (3.16)$$

where $(\mathcal{P}_{j-1}f)_k^{(j)} = \mathcal{P}_{j-1}f(x_k^{(j)})$, $-1 \leq k \leq n_j - 2$.

Again, as in step 2 we can verify that function $f_{-1}(x) + f_0(x) + \cdots + f_j(x)$ interpolates function $f(x)$ on all interpolation points $\{x_k^{(-1)}\}, \dots, \{x_k^{(j)}\}$. Especially, for $j = J$ we have $\mathcal{P}_J f(x) = f_{-1}(x) + f_0(x) + \cdots + f_J(x)$, which will satisfy the required interpolation condition (3.11).

Number of Operations.

For $j = -1$, the number of operations to invert (3.9) using Thomas algorithm to obtain $\{\hat{f}_k^{(-1)}\}$ is $5L(flops)$. For $0 \leq j \leq J$, the cost of computing the coefficients $\hat{f}_k^{(j)}$ in $f_j(x) = \mathbf{I}_{w_j}(f^{(j)} - (\mathcal{P}_{j-1}f)^{(j)}) = \sum_{-1 \leq k \leq n_j-2} \hat{f}_{j,k} \psi_{j,k}(x)$ consists of three parts: (1) evaluation of $(\mathcal{P}_{j-1}f)^{(j)} = \{\mathcal{P}_{j-1}f(x_k^{(j)})\} - (5j+7)n_j(flops)$; (2) calculating the difference $f^{(j)} - (\mathcal{P}_{j-1}f)^{(j)} - n_j(flops)$; (3) inverting the matrix \mathbf{M}_j in (3.9) - $5n_j(flops)$, totaling $(5j+13)n_j(flops)$. So the total cost of finding $\hat{\mathbf{f}} = 5L + \sum_{j=0}^J (5j+13)n_j = \sum_{j=0}^J (5j+13)2^j L \leq 6N \log N$ where again $N = 2^{J+1}L - 1$.

Now let us go back to (3.10) to see the meaning of the wavelet coefficient $\{\hat{f}_{j,k}\}$ in the finite wavelet decomposition of space $H_0^2(I)$ for function $f(x)$. For this purpose, we first take

Recalling (3.10) and the “orthogonality condition” (3.7) of the interpolation points, we have

$$\mathcal{P}_J f(x_k^{(-1)}) = f_{-1}(x_k^{(-1)}), \quad 1 \leq k \leq L-1$$

which needs $7(L-1)$ (flops).

For each $0 \leq j \leq J$, to compute $\mathcal{P}_J f(x_k^{(j)})$, $-1 \leq k \leq n_j-2$, it needs $(5j+12)*n_j$ (flops). Thus, it takes $7(L-1) + \sum_{j=0}^J (5j+12)n_j = 2^{J+1}L(5J+7) + 5L-7 \leq \mathbf{7NlogN}(flops)$ to compute the vector \mathbf{f} .

$\mathbf{f} \longrightarrow \hat{\mathbf{f}}$

Recalling that $\mathbf{f} = (\mathbf{f}^{(-1)}, \mathbf{f}^{(0)}, \dots, \mathbf{f}^{(J)})^\top$, we proceed to construction of $\mathcal{P}_J f(x)$ in the following steps.

Step 1

Define

$$f_{-1}(x) = \mathbf{I}_{V0}\mathbf{f}^{(-1)} = \hat{f}_{-1,-1}\phi_b(x) + \sum_{k=0}^{L-4} \hat{f}_{-1,k}\phi_k(x) + \hat{f}_{-1,L-3}\phi_b(L-x),$$

so $f_{-1}(x)$ interpolates $f(x)$ at the interpolation points $x_k^{(-1)}$, $-1 \leq k \leq L-1$, namely

$$f_{-1}(x_k^{(-1)}) = f(x_k^{(-1)}); \quad (3.12)$$

Step 2

Define

$$f_0(x) = \mathbf{I}_{w0}(\mathbf{f}^{(0)} - (\mathbf{I}_{V0}f)^{(0)}) = \sum_{l=-1}^{n_0-2} \hat{f}_{0,l}\psi_{0,l}(x) \quad (3.13)$$

where $(\mathbf{I}_{V0}f)^{(0)} = \{\mathbf{I}_{V0}f(x_k^{(0)})\}_{k=-1}^{n_0-2}$

As a result of the “point-wise orthogonality” conditions (3.7) of the interpolation points, we have $\psi_{0,l}(x_k^{(-1)}) = 0$, $-1 \leq l \leq n_0-2$, $1 \leq k \leq L-1$, thus

$$f_0(x_k^{(-1)}) = 0, \quad 1 \leq k \leq L-1.$$

where

$$f_{-1}(x) = \mathbf{I}_{V_0} f(x) \in V_0, \quad f_j(x) = \sum_{k=-1}^{n_j-2} \hat{f}_{j,k} \psi_{j,k}(x) \in W_j, j \geq 0,$$

and

$$\begin{aligned} \mathcal{P}_J f(x_k^{(-1)}) &= f(x_k^{(-1)}), & 1 \leq k \leq L-1 \\ \mathcal{P}_J f(x_k^{(j)}) &= f(x_k^{(j)}), & j \geq 0, -1 \leq k \leq n_j-2. \end{aligned} \quad (3.11)$$

Let us denote $\mathbf{f} = (\mathbf{f}^{(-1)}, \mathbf{f}^{(0)}, \dots, \mathbf{f}^{(J)})^\top$ the values of $f(x)$ on all interpolation points, i.e.

$$\begin{aligned} \mathbf{f}^{(-1)} &= \{f(x_k^{(-1)})\}_{k=1}^{L-1}, \\ \mathbf{f}^{(j)} &= \{f(x_k^{(j)})\}_{k=-1}^{n_j-2}, \quad j \geq 0, \end{aligned}$$

and $\hat{\mathbf{f}} = (\hat{\mathbf{f}}^{(-1)}, \hat{\mathbf{f}}^{(0)}, \dots, \hat{\mathbf{f}}^{(J)})^\top$ the wavelet coefficients in the expansion (3.10)

$$\begin{aligned} \hat{\mathbf{f}}^{(-1)} &= \{\hat{f}_{-1,k}\}_{k=1}^{L-1}, \\ \hat{\mathbf{f}}^{(j)} &= \{\hat{f}_{j,k}\}_{k=-1}^{n_j-2}, \quad j \geq 0. \end{aligned}$$

The following algorithm provides a recursive way to compute all the wavelet coefficients $\hat{\mathbf{f}}$, and also the wavelet expansion (3.10) can be expanded as needed to include higher level wavelet spaces $W_j, J+1 \leq j \leq J'$ by adding only terms from the higher wavelet spaces, i.e. $W_{J+1}, \dots, W_{J'}$.

DWT transform

$$\hat{\mathbf{f}} \longrightarrow \mathbf{f}$$

This direction of transform is straightforward by evaluating the expansion (3.10) at all the collocation points $\{x_k^{(j)}\}, j \geq -1$ to obtain \mathbf{f} . The “Point-wise Orthogonality” (3.7) of the interpolation points and the compactness of $\text{supp} \psi_{j,k}(x)$ can be used to reduce the number of evaluations.

Number of Operations

Let N be the total number of collocation points and $N = (L-1) + \sum_{j=0}^J n_j = 2^{J+1}L - 1$. In the evaluation of $\mathcal{P}_J f(x_k^{(j)})$, values of $\psi(x)$ and $\phi(x)$ at dyadic points $\frac{k}{2^j}, 0 \leq k \leq 2^j L, j \geq 0$ are needed and they can be computed once for all for future use.

This orthogonality condition will be crucial in obtaining a fast Discrete Wavelet Transform (DWT).

The interpolation $\mathbf{I}_{w_j}f(x)$ of a function $f(x) \in H_0^2(I)$ in $W_j, j \geq 0$ can be expressed as a linear combination of $\psi_{j,k}(x)$ $k = -1, \dots, n_j - 2$, namely

$$\mathbf{I}_{w_j}f(x) = \sum_{k=-1}^{n_j-2} \hat{f}_{j,k} \psi_{j,k}(x) \quad (3.8)$$

and

$$\mathbf{I}_{w_j}f(x_k^{(j)}) = f(x_k^{(j)}), \quad -1 \leq k \leq n_j - 2.$$

If we denote \mathbf{M}_j as the n_j -th order matrix which relates $\hat{\mathbf{f}}^{(j)} = (\hat{f}_{j,-1}, \dots, \hat{f}_{j,n_j-2})^\top$ and $\mathbf{f}^{(j)} = (f(x_{-1}^{(j)}), \dots, f(x_{n_j-2}^{(j)}))^\top$, then

$$\mathbf{f}^{(j)} = \mathbf{M}_j \hat{\mathbf{f}}^{(j)} \quad (3.9)$$

where

$$\mathbf{M}_j = \begin{pmatrix} 1 & -\frac{1}{14} & & & & & \\ -\frac{1}{13} & 1 & -\frac{1}{14} & & & & \\ & -\frac{1}{14} & 1 & -\frac{1}{14} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{14} & 1 & -\frac{1}{14} & \\ & & & & -\frac{1}{14} & 1 & -\frac{1}{13} \\ & & & & & -\frac{1}{14} & 1 \end{pmatrix}.$$

The solution of the coefficients $\hat{f}_{j,k}$, $-1 \leq k \leq n_j - 2$ again involves solving tridiagonal system (3.9) which costs $(5n_j)$ operations.

Now let us assume that the values of a function $f(x) \in H_0^2(I)$ are given on all the interpolation points $\{x_k^{(j)}\}$ defined in (3.1) and (3.6), we intend to find the wavelet interpolation $\mathcal{P}_J f(x) \in V_0 \oplus W_0 \oplus W_1 \cdots \oplus W_J$ for $J \geq 0$, i.e.

$$\begin{aligned} \mathcal{P}_J f(x) &= \hat{f}_{-1,-1} \phi_b(x) + \sum_{k=0}^{L-4} \hat{f}_{-1,k} \phi_k(x) + \hat{f}_{-1,L-3} \phi_b(L-x) \\ &+ \sum_{j=0}^J \left[\sum_{k=-1}^{n_j-2} \hat{f}_{j,k} \psi_{j,k}(x) \right] \\ &= f_{-1}(x) + \sum_{j=0}^J f_j(x) \end{aligned} \quad (3.10)$$

The cubic interpolant $\mathbf{I}_{V_0}f(x)$ of data $\{f_k^{(-1)}\}$ can be expressed as follows,

$$\mathbf{I}_{V_0}f(x) = c_{-1}\phi_b(x) + \sum_{k=0}^{L-4} c_k\phi_{0,k}(x) + c_{L-3}\phi_b(L-x) \quad (3.3)$$

and $\mathbf{I}_{V_0}f(x)$ interpolates data $f_k^{(-1)}$, $k = 1, \dots, L-1$, namely

$$\mathbf{I}_{V_0}f(x_k^{(-1)}) = f_k^{(-1)}, \quad k = 1, \dots, L-1. \quad (3.4)$$

Let \mathbf{B} be the transform matrix between $\mathbf{f}^{(-1)} = (f_1^{(-1)}, \dots, f_{L-1}^{(-1)})^\top$ and the coefficient $\mathbf{c} = (c_{-1}, \dots, c_{L-3})^\top$, i.e.

$$\mathbf{f}^{(-1)} = \mathbf{B}\mathbf{c} \quad (3.5)$$

where

$$\mathbf{B} = \begin{pmatrix} \frac{7}{12} & \frac{1}{6} & & & & & \\ \frac{1}{6} & \frac{2}{3} & & & & & \\ & \frac{1}{6} & \frac{2}{3} & & & & \\ & & \frac{1}{6} & \frac{2}{3} & & & \\ & & & \frac{1}{6} & & & \\ & & & & \ddots & & \\ & & & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & & & \frac{2}{3} & \frac{1}{6} & \frac{7}{12} \\ & & & & & & \frac{1}{6} & \frac{7}{12} \end{pmatrix}.$$

In order to obtain the coefficients c_k , $-1 \leq k \leq L-3$ in (3.3), we have to solve the triadiagonal system (3.5) which involves $(5L)$ operations.

Interpolation Operator $\mathbf{I}_{w_j}f$ in W_j

Similarly, we can define the interpolation operator $\mathbf{I}_{w_j}f(x)$ in W_j , $j \geq 0$ for any function $f(x)$ in $H_0^2(I)$. For this purpose, we choose the following interpolation points in I ,

$$x_k^{(j)} = \frac{k+1.5}{2^j}, \quad -1 \leq k \leq n_j - 2 \quad (3.6)$$

where $n_j = \dim W_j = 2^j L$.

It can be easily checked that the interpolation points $\{x_k^{(-1)}\}$ for V_0 in (3.1) and $\{x_k^{(j)}\}$ for W_j , $j \geq 0$ in (3.6) satisfy a “point-wise orthogonality” condition.

Point Orthogonality of $\{x_k^{(j)}\}$ for $j > i$, $-1 \leq k \leq n_j - 2$,

$$\begin{aligned} \psi_{j,k}(x_k^{(j)}) &= 1 \\ \psi_{j,k}(x_l^{(i)}) &= 0, \quad -1 \leq \ell \leq n_i - 2 \text{ if } i \geq 0; \quad 1 \leq \ell \leq L-1 \text{ if } i = -1. \end{aligned} \quad (3.7)$$

$$c_2 = -\frac{3}{2}, \quad c_3 = \frac{1}{3}.$$

then $s = 3$ in (2.37), and thus, equation (2.35) is satisfied within an error of $O(h^3)$. Correspondingly, the coefficients $\alpha_k, 1 \leq k \leq 4$ for $\mathbf{I}_{b,j}f(x)$ become

$$\begin{aligned} \alpha_1 &= \sum_{k=0}^p c'_k f(kh), & \alpha_2 &= f(0) \\ \alpha_3 &= -\sum_{k=0}^p c'_k f(L - kh), & \alpha_4 &= f(L) \end{aligned} \quad (2.38)$$

where

$$c'_0 = \left(\frac{1}{2^{j+1}h}c_0 - \frac{3}{2}\right), \quad c'_k = \frac{c_k}{2^{j+1}h}, \quad 1 \leq k \leq p.$$

Now we have $f(x) - \mathbf{I}_{b,j}f(x) \in H_0^2(I)$ and the decomposition (2.30) can be applied for it. Therefore, we can find an approximation $f_j(x)$ for any function $f(x) \in H^2(I)$ as close as possible, provided that j is large enough, in the form of

$$f_j(x) = \mathbf{I}_{b,j}f + f_0 + g_0 + g_1 \cdots + g_j \quad (2.39)$$

where $f_0(x) \in V_0, g_i \in W_i, 0 \leq i \leq j$.

3 Discrete Wavelet Transform (DWT)

In this section, we will introduce a fast Discrete Wavelet Transform (DWT) which maps discrete sample values of a function to its wavelet interpolant expansions. Such expansion with the wavelet decomposition will enable us to compute an approximation of the derivatives of the function.

Interpolant Operator \mathbf{I}_{V_0} in V_0

Consider any function $f(x) \in H_0^2(I)$ and denote the interior knots for V_0 by

$$x_k^{(-1)} = k, \quad k = 1, \dots, L-1 \quad (3.1)$$

and the values of $f(x)$ on $\{x_k^{(-1)}\}_{k=1}^{L-1}$ by

$$f_k^{(-1)} = f(x_k^{(-1)}), \quad k = 1, \dots, L-1. \quad (3.2)$$

Approximation for function in $H^2(I)$

Consider the following two functions

$$\eta_1(x) = 2x_+ - 3x_+^2 + \frac{7}{6}x_+^3 - \frac{4}{3}(x-1)_+^3 \quad (2.31)$$

$$\eta_2(x) = (1-x)_+^3. \quad (2.32)$$

For any function $f(x) \in H^2(I)$, by the Sobolev embedding theorem we have $f(x) \in C^1(I)$ and, therefore, we can define the following boundary interpolation $\mathbf{I}_{b,j}f(x)$, $j \geq 0$

$$\mathbf{I}_{b,j}f(x) = \alpha_1\eta_1(2^jx) + \alpha_2\eta_2(2^jx) + \alpha_3\eta_1(2^j(L-x)) + \alpha_4\eta_2(2^j(L-x)) \quad (2.33)$$

such that

$$\mathbf{I}_{b,j}f(0) = f(0), \quad \mathbf{I}_{b,j}f(L) = f(L) \quad (2.34)$$

$$\mathbf{I}_{b,j}f'(0) = f'(0), \quad \mathbf{I}_{b,j}f'(L) = f'(L). \quad (2.35)$$

It can be easily verified that, in order to have $\mathbf{I}_{b,j}f$ satisfy conditions (2.34) - (2.35), we have to take

$$\begin{aligned} \alpha_1 &= \frac{f'(0)}{2^{j+1}} - \frac{3}{2}f(0), & \alpha_2 &= f(0) \\ \alpha_3 &= -\frac{f'(L)}{2^{j+1}} - \frac{3}{2}f(L), & \alpha_4 &= f(L). \end{aligned} \quad (2.36)$$

In many situations we do not have the values of derivatives $f'(0), f'(L)$. However, they can be approximated by finite differences using only the values of $f(x)$. To preserve the correct order of accuracy for a cubic spline approximation, we suggest using the following approximations

$$\begin{aligned} f'(0) &= \frac{1}{h} \sum_{k=0}^p c_k f(kh) + O(h^s) \\ f'(L) &= -\frac{1}{h} \sum_{k=0}^p c_k f(L - kh) + O(h^s). \end{aligned} \quad (2.37)$$

where $h > 0$ and $p \geq 3$. For $p = 3$, if we take

$$c_0 = -\frac{11}{6}, \quad c_1 = 3$$

Using equation (2.21) and the identity

$$\phi^{(4)}(x) = \frac{3}{4} \sum_{j=0}^4 \binom{4}{j} (-1)^j \delta(x-j)$$

where $\delta(x)$ is the Dirac- δ function, so we have

$$\langle \phi(x-l), \psi(x-k) \rangle = \frac{3}{4} \sum_{j=0}^4 \binom{4}{j} (-1)^j \psi(j-(k-l)) = 0.$$

Equations (2.27) - (2.29) can be shown similarly to be true . So (1) follows from (2.19) and (2.20) and the fact that $\dim V_j = 2^j L - 3$ and $\dim W_j = 2^j L$ and $\dim V_{j+1} = 2^{j+1} L - 3 = (2^j L - 3) + 2^j L = \dim V_j + \dim W_j$;

(2) follows from (1);

(3) follows directly from **Theorem 1** (ii).

□

As a consequence of Theorem 2, any function $f(x) \in H_0^2(I)$ can be approximated as closely as possible by a function $f_j(x) \in V_j = V_0 \oplus W_0 \oplus W_1 \cdots \oplus W_j$ for a sufficiently large j , and $f_j(x)$ has a unique orthogonal decomposition

$$f_j(x) = f_0 + g_0 + g_1 + \cdots + g_j \tag{2.30}$$

where $f_0 \in V_0, g_i \in W_i, 0 \leq i \leq j$.

$$\psi_{b,j}^l(x) = \psi_b(2^j x), \quad \psi_{b,j}^r(x) = \psi_b(2^j(L - x)) \quad (2.23)$$

where $n_j = 2^j L$. For the sake of simplicity, we will adopt the following notations

$$\psi_{j,-1}(x) = \psi_{b,j}^l(x), \quad \psi_{j,n_j-2}(x) = \psi_{b,j}^r(x). \quad (2.24)$$

So when $k = -1, n_j - 2$, $\psi_{j,k}(x)$ will denote the two boundary wavelet functions, not the usual translation and dilation of $\psi(x)$.

Finally, for each $j \geq 0$, we define

$$W_j = \text{clos}_{H_0^2} \langle \psi_{j,k}(x), k = -1, \dots, n_j - 2 \rangle. \quad (2.25)$$

Theorem 2 The $W_j, j \geq 0$ defined in (2.25) is the orthogonal complement of V_j in V_{j+1} under the inner product (2.3), i.e.

(1) $V_{j+1} = V_j \oplus W_j$ for $j \in \mathbf{Z}^+$. Here \oplus stands for $V_k \perp W_j$ under the inner product (2.3) and $V_{j+1} = V_j + W_j$. Therefore,

(2) $W_j \perp W_{j+1}, j \in \mathbf{Z}^+$;

(3) $H_0^2(I) = V_0 \oplus_{j \in \mathbf{Z}^+} W_j$.

Proof. (1) We only have to prove $V_j \perp W_j$ for $j = 0$, namely, for $0 \leq l \leq L-4, 0 \leq k \leq L-3$,

$$\langle \phi(x-l), \psi(x-k) \rangle = 0 \quad (2.26)$$

$$\langle \phi(x-l), \psi_b(x) \rangle = 0 \quad (2.27)$$

$$\langle \phi_b(x), \psi(x-k) \rangle = 0 \quad (2.28)$$

$$\langle \phi_b(x), \psi_b(x) \rangle = 0. \quad (2.29)$$

Integrating by parts twice in (2.26) and using the fact that $\psi(x), \phi(x) \in H_0^2(I)$, we have

$$\begin{aligned} \langle \phi(x-l), \psi(x-k) \rangle &= \int_0^L \phi''(x-l) \psi''(x-k) dx \\ &= \phi''(x-l) \psi'(x-k) \Big|_0^L - \int_0^L \phi^{(3)}(x-l) \psi'(x-k) dx \\ &= - \int_0^L \phi^{(3)}(x-l) \psi'(x-k) dx \\ &= -\phi^{(3)}(x-l) \psi(x-k) \Big|_0^L + \int_0^L \phi^{(4)}(x-l) \psi(x-k) dx \\ &= \int_0^L \phi^{(4)}(x-l) \psi(x-k) dx. \end{aligned}$$

the partition π interpolating $f(x)$. From the fact that $f(0) = f(L) = f'(0) = f'(L) = 0$, we have $s(x)$ in V_j and then

$$s(x) = c_{-1}\phi_{b,j}(x) + \sum_{k=0}^{L'-4} c_k\phi_{j,k}(x) + c_{L-3}\phi_{b,j}(L-x) \quad (2.17)$$

such that

$$s(x_i) = f(x_i), \quad 0 \leq i \leq 2^j L \quad (2.18)$$

where $L' = 2^j L$, $x_i = \frac{i}{2^j}$.

Finally, from (2.16) in Lemma 3 with $r = 2$ we have

$$|||s - f||| = ||s^{(2)} - f^{(2)}|| \leq \epsilon_2 ||f^{(4)}|| / 2^{2j}.$$

Therefore, as $j \rightarrow \infty$, $|||s - f||| \rightarrow 0$. This proves that $C_0^\infty(0, L) \subset \text{clos}_{H_0^2}(\bigcup_{j \in \mathbf{Z}^+} V_j)$.

Then, Theorem 1 (ii) follows from the fact that $C_0^\infty(0, L)$ is dense in $H_0^2(0, L)$.

□

To construct a wavelet decomposition of Sobolev space $H_0^2(I)$ under the inner product (2.3), we consider the following two wavelet functions $\psi(x), \psi_b(x)$ (see Figure 2) ,

$$\psi(x) = -\frac{3}{7}\phi(2x) + \frac{12}{7}\phi(2x-1) - \frac{3}{7}\phi(2x-2) \in V_1 \quad (2.19)$$

$$\psi_b(x) = \frac{24}{13}\phi_b(2x) - \frac{6}{13}\phi(2x) \in V_1. \quad (2.20)$$

It can be verified that

$$\psi(n) = \psi_b(n) = 0, \text{ for all } n \in \mathbf{Z}. \quad (2.21)$$

Equation (2.21) will be important in the construction of the fast DWT transform later. And, equations (2.19) and (2.20) imply that $\psi(x)$ and $\psi_b(x)$ both belong to V_1 . As usual, we define the dilation and translation of these two functions

$$\psi_{j,k}(x) = \psi(2^j x - k), \quad j \geq 0, \quad k = 0, \dots, n_j - 3, \quad (2.22)$$

Theorem 1 Let $V_j, j \in \mathbf{Z}^+$ be the linear span of (2.14), then V_j forms a multiresolution analysis (MRA) for $H_0^2(I)$ equipped with norm (2.4) in the following sense

(i) $V_0 \subset V_1 \subset V_2 \subset \cdots$;

(ii) $\text{clos}_{H_0^2}(\bigcup_{j \in \mathbf{Z}^+} V_j) = H_0^2(I)$;

(iii) $\bigcap_{j \in \mathbf{Z}^+} V_j = V_0$; and

(iv) for each j , $\{\phi_{j,k}(x) = \phi(2^j x - k), \phi_{b,j}(x) = \phi_b(2^j x), \phi_{b,j}(L - x)\}$ is an unconditional basis of V_j .

Proof. The proof for (iii) and (iv) is straightforward and omitted here. The proof for (i) follows from (2.7) in Lemma 1. In order to prove (ii), we recall a familiar result on interpolation cubic spline approximation for smooth functions taken from [14] and rewritten for the proof of our theorem.

□

Lemma 3 Let π be the partition given by $x_i = ih, 0 \leq i \leq n, h = \frac{(b-a)}{n}$ and $s(x)$ be the cubic spline interpolating $f(x) \in C^4[a, b]$ at all points in π ,

$$s(x_i) = f(x_i), \quad 0 \leq i \leq n,$$

and satisfying the following boundary conditions:

$$s'(a) = f'(a), \quad s'(b) = f'(b). \quad (2.15)$$

Then $s(x)$ uniquely exists and

$$\|s^{(r)} - f^{(r)}\|_{L^2} \leq \epsilon_r \|f^{(4)}\|_{L^2} h^{4-r}, \quad r = 0, 1, 2, 3 \quad (2.16)$$

where $\epsilon_0 = \frac{5}{384}, \epsilon_1 = \frac{1}{24}, \epsilon_2 = \frac{3}{8}, \epsilon_3 = 1$.

Proof of (ii) of Theorem 1 Let $h = \frac{1}{2^j}, a = 0$ and $b = L$. Consider $f(x) \in C_0^\infty(0, L)$, Since $C_0^\infty(0, L) \subset C^4[0, L] \cap H_0^2(0, L)$, by Lemma 3, there is an unique cubic spline corresponding

1)

$$\phi(x) = N_4(x) = \frac{1}{6} \sum_{j=0}^4 \binom{4}{j} (-1)^j (x-j)_+^3 \quad (2.5)$$

$$\phi_b(x) = \frac{3}{2}x_+^2 - \frac{11}{12}x_+^3 + \frac{3}{2}(x-1)_+^3 - \frac{3}{4}(x-2)_+^3 \quad (2.6)$$

where $N_4(x)$ is the 4th order B-spline [13] and for any real number n

$$x_+^n = \begin{cases} x^n & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In a pair they satisfy the following two-scale relationship,

Lemma 1

$$\begin{aligned} \phi(x) &= \sum_{k=0}^4 2^3 \binom{4}{k} \phi(2x-k) \\ \phi_b(x) &= \beta_{-1} \phi_b(2x) + \sum_{k=0}^2 \beta_k \phi(2x-k) \\ \text{here } \beta_{-1} &= \frac{3}{4}, \beta_0 = -\frac{3}{8}, \beta_1 = \frac{17}{4}, \beta_2 = -\frac{13}{4}. \end{aligned} \quad (2.7)$$

We summarize some properties of $\phi(x)$ and $\phi_b(x)$ in the following lemma.

Lemma 2 Let $\phi(x)$ and $\phi_b(x)$ be defined as in (2.5) and (2.6), then we have

$$(1) \quad \text{supp}(\phi(x)) = [0, 4]; \quad (2.8)$$

$$(2) \quad \text{supp}(\phi_b(x)) = [0, 3]; \quad (2.9)$$

$$(3) \quad \phi(x), \phi_b(x) \in H_0^2(I); \quad (2.10)$$

$$(4) \quad \phi'(1) = -\phi'(3) = \frac{1}{2}, \phi'(2) = 0, \phi'_b(1) = \frac{1}{4}, \phi'_b(2) = -\frac{1}{2}; \quad (2.11)$$

$$(5) \quad \phi(1) = \phi(3) = \frac{1}{6}, \phi(2) = \frac{2}{3}, \phi_b(1) = \frac{7}{12}, \phi_b(2) = \frac{1}{6}. \quad (2.12)$$

For any $j, k \in \mathbb{Z}$, we define

$$\phi_{j,k}(x) = \phi(2^j x - k), \quad \phi_{b,j}(x) = \phi_b(2^j x). \quad (2.13)$$

And for each j , let V_j be the closure under norm $|||f|||$ in (2.4) of the linear span of $\{\phi_{j,k}(x), 0 \leq k \leq 2^j L - 4, \phi_{b,j}(x), \phi_{b,j}(L - x)\}$, namely

$$V_j = \text{span}\{\phi_{j,k}(x), 0 \leq k \leq 2^j L - 4, \phi_{b,j}(x), \phi_{b,j}(L - x)\}. \quad (2.14)$$

olution analysis (MRA) and its corresponding wavelet decomposition of the Sobolev space $H_0^2(I)$ are constructed using $\phi(x), \phi_b(x)$ and $\psi(x), \psi_b(x)$. Then, we show how to construct a wavelet approximation for function in Sobolev space $H^2(I)$ which the solutions of PDE's will belong to. In Section 3, we discuss the fast discrete wavelet transform (DWT) between functions and their wavelet coefficients. In Section 4, we discuss the derivative matrix \mathcal{D} associated with wavelet interpolations. In Section 5, we present the wavelet collocation methods for nonlinear time evolution PDE's. In Section 6, we give the CPU time performance of the DWT transforms and the numerical results of the wavelet collocation methods for linear and nonlinear PDE's, and a conclusion is given in Section 7.

2 Scaling functions $\phi(x), \phi_b(x)$ and wavelet functions $\psi(x), \psi_b(x)$

Let I denote any finite interval, say $I = [0, L]$ and L is a positive integer (for the sake of simplicity, we assume that $L > 4$), and $H^2(I)$ and $H_0^2(I)$ denote the following two Sobolev spaces with finite L^2 norm for up to the second derivatives, i.e.

$$H^2(I) = \{f(x), x \in I \mid \|f^{(i)}\|_2 < \infty, i = 0, 1, 2\} \quad (2.1)$$

$$H_0^2(I) = \{f(x) \in H^2(I) \mid f(0) = f'(0) = f(L) = f'(L) = 0\}. \quad (2.2)$$

It can be easily checked [7] that $H_0^2(I)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_I f''(x)g''(x) dx, \quad (2.3)$$

thus,

$$|||f||| = \sqrt{\langle f, f \rangle} \quad (2.4)$$

provides a norm for $H_0^2(I)$.

In order to generate a multiresolution for Sobolev space $H_0^2(I)$, we consider two scaling functions, an interior scaling function $\phi(x)$ and a boundary scaling function $\phi_b(x)$ (see Figure

1 Introduction

Wavelet approximations have attracted much attention as a potential efficient numerical technique for the solutions of partial differential equations [1] - [6]. Because of their advantageous properties of localizations in both space and frequency domains [8] - [10], wavelets seem to be a great candidate for adaptive schemes for solutions which vary dramatically both in space and time and develop singularities. However, in order to take advantage of the nice properties of wavelet approximations, we have to find an efficient way to deal with the nonlinearity and general boundary conditions in the PDE's. After all, most of the problems of fluid dynamics and electromagnetics, which involve solutions with quite different scales, are governed by nonlinear PDE's with complicated boundary conditions. Therefore, it is our objective here to address these issues when designing wavelet approximations and numerical schemes for nonlinear PDE's.

We will present a new wavelet collocation method designed to solve nonlinear time evolution problems. The key component in this collocation method is a so-called "Discrete Wavelet Transform" (DWT) which maps a solution between the physical space and the wavelet coefficient space. The wavelet decomposition is based on a new cubic spline wavelet for $H_0^2(I)$ where I is a bounded interval [11]. In order to treat the boundary conditions an extra boundary scaling function $\phi_b(x)$ and a boundary wavelet $\psi_b(x)$ have been used. A special "pointwise orthogonality" (see(3.7)) of the wavelet functions $\psi_{j,k}(x)$ results in $O(N \log N)$ operations for the DWT transform where N is the total number of unknowns. Therefore, the nonlinear term in the PDE can be easily treated in the physical space, and the derivatives of those nonlinear terms then computed in the wavelet space. As a result, collocation methods will provide the flexibility of handling nonlinearity (and, also the implementation of various boundary conditions) which usually are not shared by Galerkin type wavelet methods and finite element methods.

The rest of this paper is divided into the following five sections. In section 2, we introduce the cubic scaling functions $\phi(x)$, $\phi_b(x)$ and their wavelet functions $\psi(x)$, $\psi_b(x)$. A multires-

ADAPTIVE WAVELET COLLOCATION METHODS FOR INITIAL VALUE BOUNDARY PROBLEMS OF NONLINEAR PDE'S

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ABSTRACT

We have designed a cubic spline wavelet decomposition for the Sobolev space $H_0^2(I)$ where I is a bounded interval. Based on a special “point-wise orthogonality” of the wavelet basis functions, a fast Discrete Wavelet Transform (DWT) is constructed. This DWT transform will map discrete samples of a function to its wavelet expansion coefficients in $O(N \log N)$ operations. Using this transform, we propose a collocation method for the initial value boundary problem of nonlinear PDE's. Then, we test the efficiency of the DWT transform and apply the collocation method to solve linear and nonlinear PDE's.

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